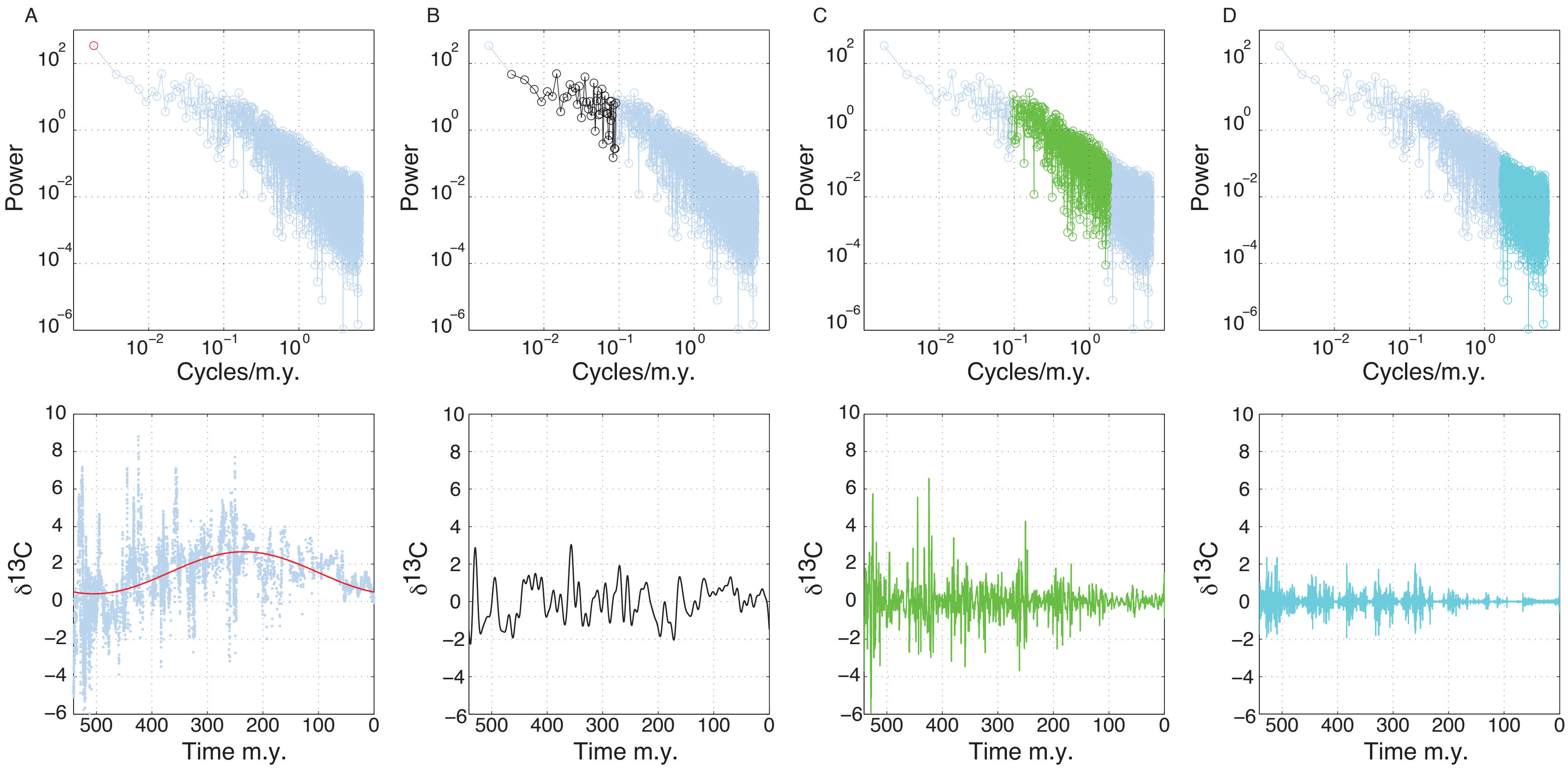

Figure S1: Spectral decomposition of the Phanerozoic record.

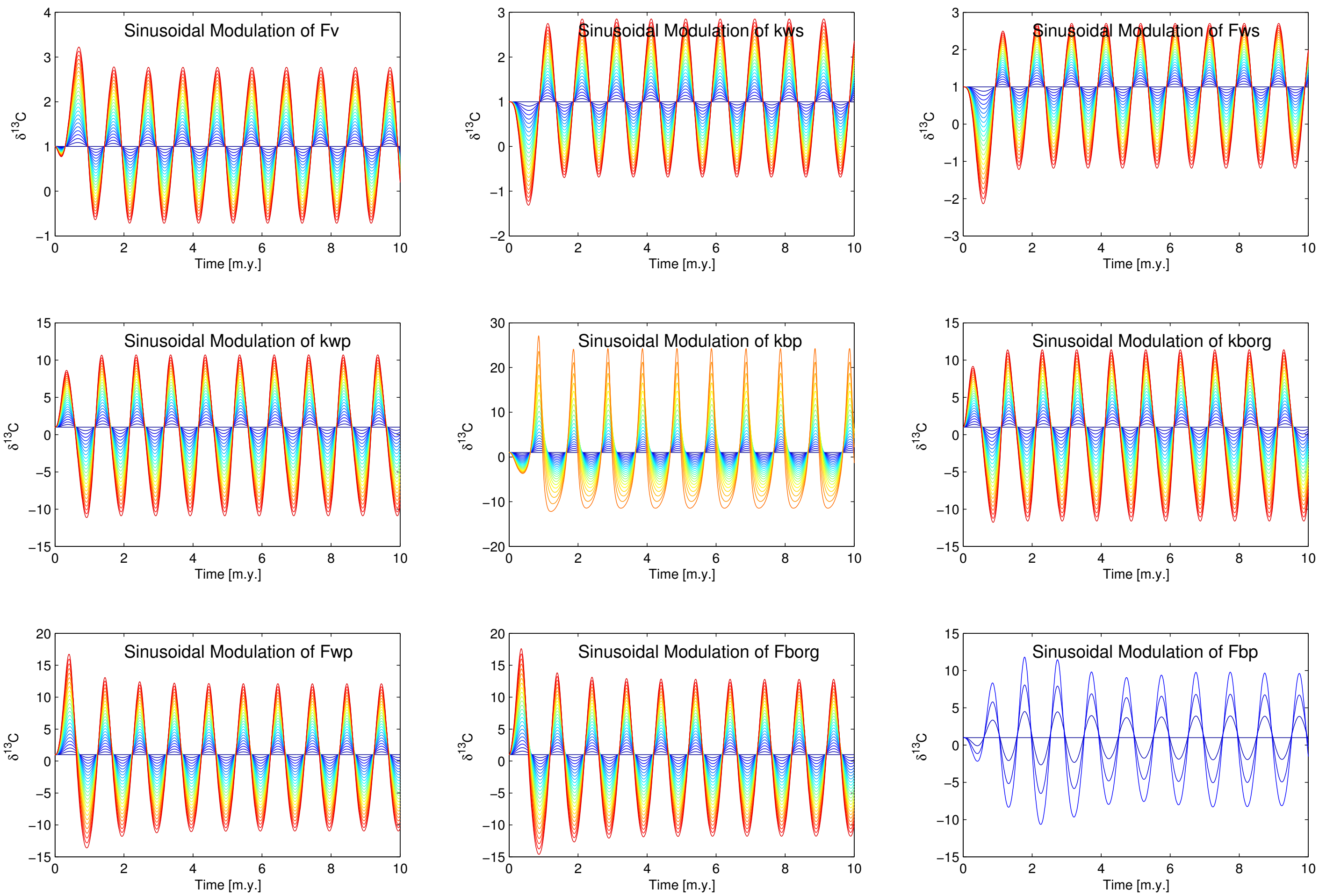

Figure S2: Model δ^{13}C response to sinusoidal modulation of parameters with a period of 1 m.y. The flux being modulated in each instance is denoted by the name of the flux in the upper part of the subplot (e.g. Fv). Modulation of a flux coefficient is denoted by the name of the rate parameter (e.g. kwp). The resulting δ^{13}C output is given as a function of time. Frequency and amplitude combinations resulting in non-physical reservoir values (negative mass) were not plotted.

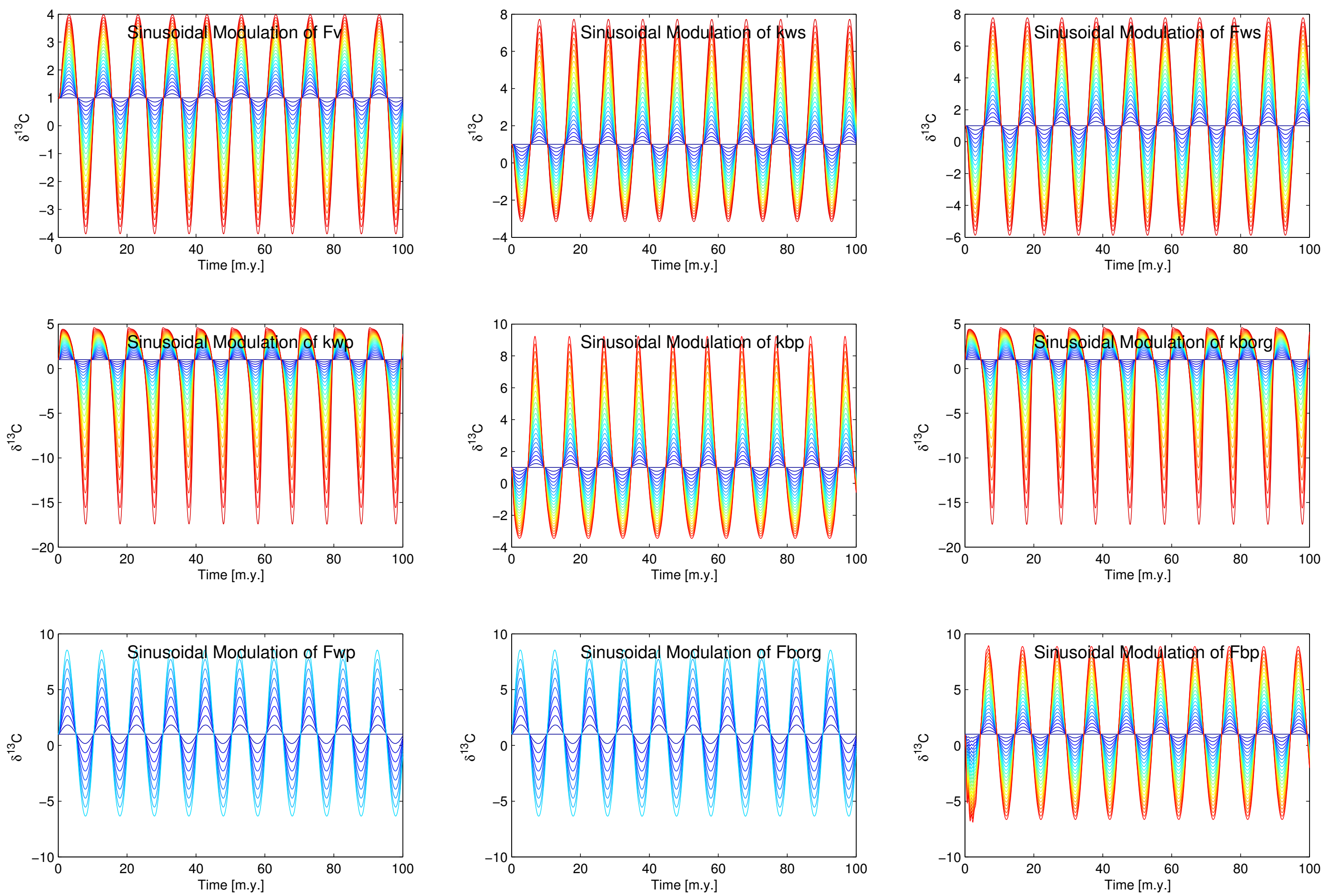

Figure S3: As in previous figure but with a period of the forcing of 10 m.y.

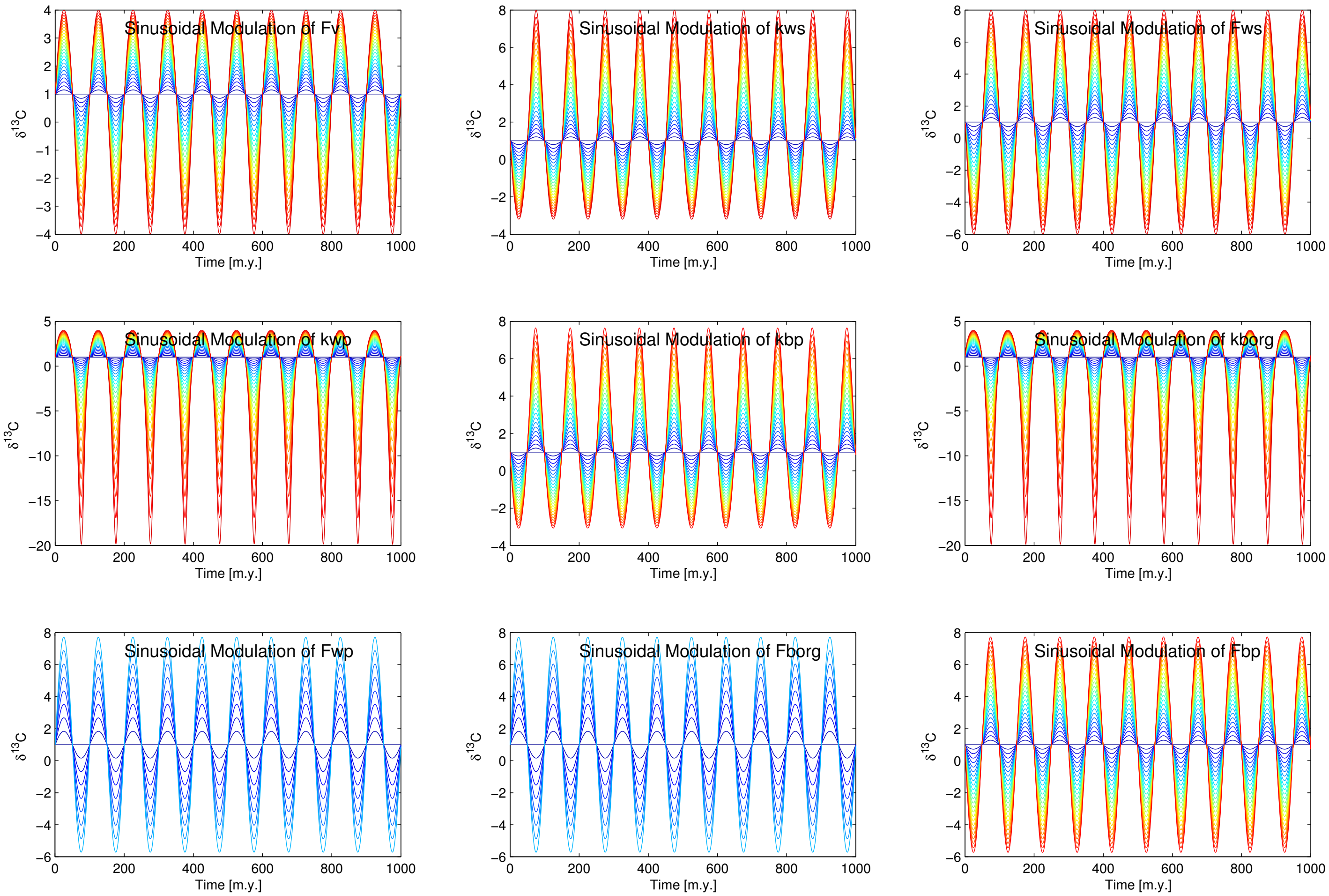

Figure S4: As in previous figure but with a period of the forcing of 100 m.y.

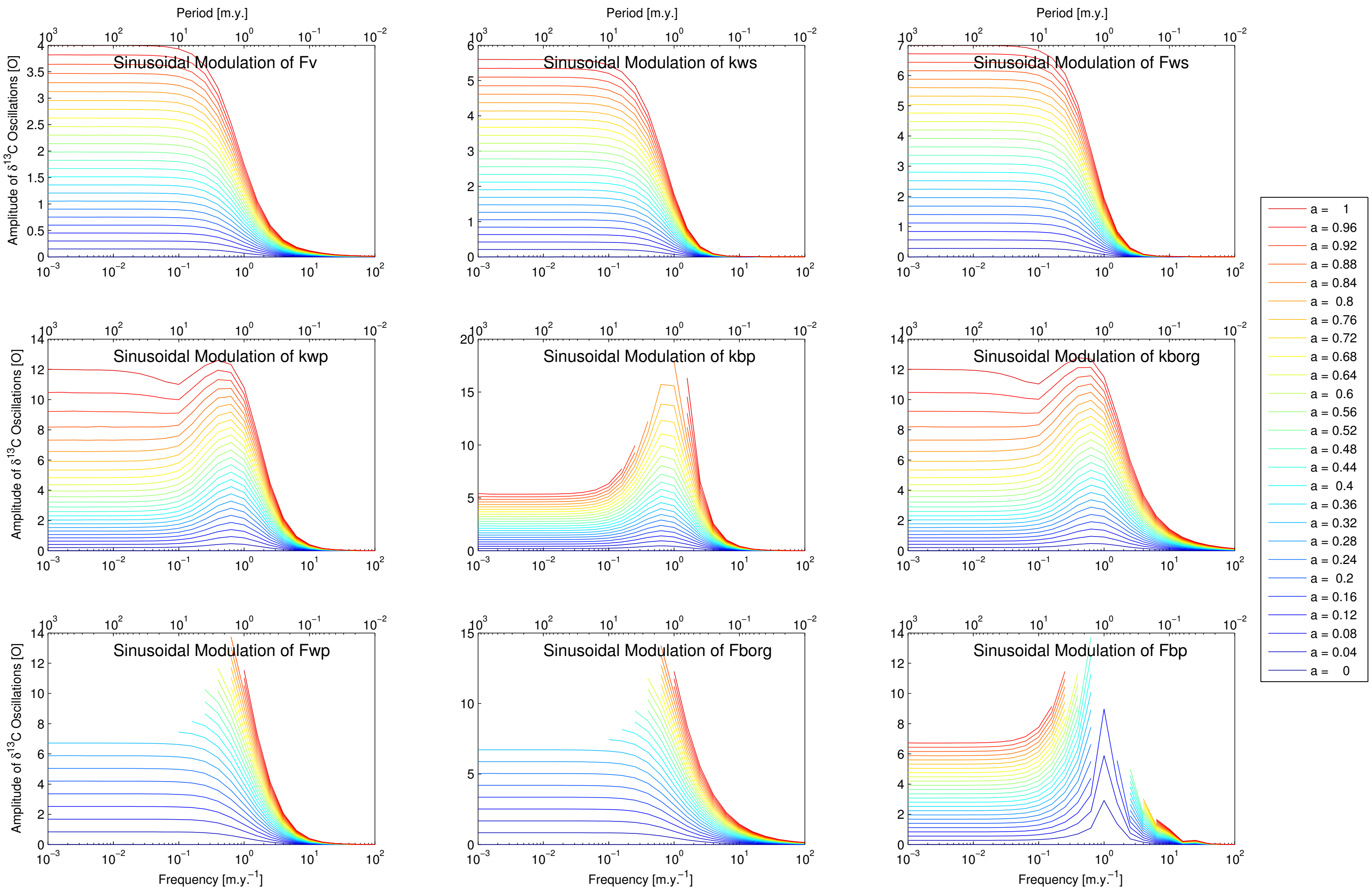

Figure S5: Model δ^{13}C frequency response to sinusoidal modulation of parameters. Modulation of a flux is denoted by the name of the flux (e.g. Fv). Modulation of a flux sensitivity is denoted by the name of the rate parameter (e.g. kwp). Amplitude of the resulting δ^{13}C oscillations on the y axis, frequency on the lower x-axis, and period of the forcing on the upper x-axis. Frequency and amplitude combinations resulting in non-physical reservoir values (negative mass) were not plotted, hence the truncation of some lines.

4 Descriptive statistics of Phanerozoic δ^{13}C Data

Period	Duration [M.yr.]	Number of unique data points	Average data spacing [yr]	Data density [points/M.yr.]
Cenozoic	66.18	27259	4125	242
Cenozoic (downsampled)	66.18	481	135,536	7.4
Cretaceous	80.18	229	350,162	1.6
Jurassic	54.51	238	229,035	4.3
Triassic	50.58	595	85,479	11.76
Permian	46.72	414	113,123	8.8
Carboniferous	60.05	1133	53,000	18.9
Devonian	60.25	900	66,945	14.9
Silurian	24.51	626	39,345	24.5
Ordovician	44.09	673	67,101	14.9
Cambrian	55.63	1984	28,053	35.6
Total	542.7	6564	119845	14.1

5 Model Values

Variable Name	Steady State Value [mol] or [mol yr^{-1}]
M_p	2x10^{15}
M_c	3.8x10^{18}
F_{wp}	3.6x10^{10}
F_{bp}	3.6x10^{10}
F_v	4x10^{12}
F_{ws}	4x10^{12}
F_{bo}	12x10^{12}
F_{wo}	12x10^{12}
δ	

Coefficient Name	Calculated via	Steady State Value [yr^{-1}]
k_{wp}	F_{wp}/M_c	9.4737x10^{-9}
k_{bp}	F_{bp}/M_p	1.8x10^{-5}
k_{bo}	F_{bo}/M_p	6x10^{-3}
k_{ws}	F_{ws}/M_c	1.0526x10^{-6}

6 Derivation of Condition for Oscillations

The relationship between the coefficients and the behavior of the system can be formally established. (See the appendix of [1], for a similar development in a climate modelling context. See [2], and [3, 4], for related mathematical treatments in a geochemical context. See [5], for an approachable introduction to linear systems from a control theory standpoint and [6], for an introduction with a mechanical and electrical engineering flavor.)

To establish the condition for oscillations we write our system of equations in matrix form:

$$\begin{bmatrix} \dot{M}_p \\ \dot{M}_c \end{bmatrix} = \begin{bmatrix} -k_{bp} & k_{wp} \\ -k_{bo} & -k_{ws} \end{bmatrix} \begin{bmatrix} M_p \\ M_c \end{bmatrix} + \begin{bmatrix} 0 \\ F_v + F_{wo} \end{bmatrix} \tag{1}$$

Or more compactly:

$$\dot{\vec{M}} = \underline{K}\vec{M} + \vec{F} \tag{2}$$

We use arrows to indicate vectors and underlining to indicate matrices. The behavior of the system is determined by the eigenvalues of the coefficient matrix:

$$\underline{K} = \begin{bmatrix} -k_{bp} & k_{wp} \\ -k_{bo} & -k_{ws} \end{bmatrix} \tag{3}$$

These can be calculated via the eigenvalue equation:

$$\underline{K}\vec{x} = \lambda\vec{x} \tag{4}$$

A nontrivial solution exists only when the determinant of $\lambda \underline{I} - \underline{K}$ equals zero:

$$\begin{vmatrix} \lambda + k_{bp} & -k_{wp} \\ k_{bo} & \lambda + k_{ws} \end{vmatrix} = (\lambda + k_{bp})(\lambda + k_{ws}) + k_{wp}\, k_{bo} = 0 \tag{5}$$

The resulting characteristic polynomial is:

$$\lambda^2 + (k_{bp} + k_{ws})\,\lambda + k_{bp}\, k_{ws} + k_{wp}\, k_{bo} = 0 \tag{6}$$

Whose roots are given by:

$$\lambda_{1,2} = \frac{-(k_{bp} + k_{ws}) \pm \sqrt{(k_{bp} + k_{ws})^2 - 4\,(k_{bp}\, k_{ws} + k_{wp}\, k_{bo})}}{2} \tag{7}$$

Oscillatory solutions (complex eigenvalues) exist when the discriminant is less than zero:

$$(k_{bp} + k_{ws})^2 - 4\,(k_{bp}\, k_{ws} + k_{wp}\, k_{bo}) < 0 \tag{8}$$

Gathering terms gives the condition for oscillations.

$$(k_{bp} - k_{ws})^2 < 4\,(k_{wp}\, k_{bo}) \tag{9}$$

6.1 Comparison with a Mass-Spring Harmonic Oscillator

In a mass-spring system the forces in operation are the force exerted on the mass by the spring, and the friction operating on the mass. We assume that the force exerted by the spring is linearly related to the displacement of the spring (Hook's law), and the damping is linearly dependent on the velocity of the mass. The sum of the two forces are equal to the change in momentum as expressed by $F = ma$, or in its differential form:

$$-\,kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2} \tag{10}$$

Since there is an equivalency between a large mass on a large spring and a small mass on a small spring, it is convenient to introduce non-dimensional variables, the angular frequency and the damping ratio:

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \zeta = \frac{c}{2\sqrt{mk}} \tag{11}$$

The equation of motion then becomes:

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = 0 \tag{12}$$

This equation can be converted to two first order differential equations by defining x_1, position, and x_2, velocity:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -2\zeta\omega_0 x_2 - \omega_0^2 x_1 \end{aligned} \tag{13}$$

Writing the system in matrix form reveals the similarity to the carbon cycle oscillator model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{14}$$

Here the "cross" terms are 1 and $-\omega_0^2$. The characteristic polynomial is:

$$\begin{vmatrix} \lambda & -1 \\ \omega_0^2 & \lambda + 2\zeta\omega_0 \end{vmatrix} = \lambda^2 + 2\zeta\omega_0\lambda + \omega_0^2 \tag{15}$$

Oscillatory solutions (complex eigenvalues) exist when the discriminant is less than zero:

$$4\,\omega_0^2\,(\zeta^2 - 1) < 0 \tag{16}$$

Hence, when $\zeta < 1$ the system will be underdamped and oscillate, whereas when $\zeta > 1$ the system will exhibit smoothly decaying solutions.

References

[1] Pollard, D, Kump, L, & Zachos, J. (2013) Interactions between carbon dioxide, climate, weathering, and the Antarctic ice sheet in the earliest Oligocene. *Global and Planetary Change* **111**, 258–267.

[2] Southam, J. R & Hay, W. W. (1976) Dynamical formulation of Broecker's model for marine cycles of biologically incorporated elements. *Journal of the International Association for Mathematical Geology* **8**, 511–527.

[3] Lasaga, A. C. (1980) The kinetic treatment of geochemical cycles. *Geochimica et Cosmochimica Acta* **44**, 815–828.

[4] Lasaga, A. C. (1981) *Dynamic treatment of geochemical cycles: global kinetics*, eds. Lasaga, A & Kirkpatrick, R. Vol. 8, pp. 69–110.

[5] Aström, K. J & Murray, R. M. (2010) *Feedback systems: an introduction for scientists and engineers.* (Princeton university press).

[6] Lathi, B. (1992) *Linear systems and signals.* (Berkeley-Cambridge Press).